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ON THE CONSERVATION OF FINITE-DIFFERENCE ABSOLUTE VORTICITY IN BAROTROPIC FORECASTS

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ABSTRACT

In solving the differential equations used in numerical weather prediction, one resorts to the use of finite-difference techniques. In the barotropic model it is assumed that the absolute vorticity is conserved. Even though the differential form of the vorticity equation insures conservation, the finite-difference form may not. The purpose of this study is to test this assumption in order to find out how serious truncation errors really are. The results indicate that the finite-difference absolute vorticity is reasonably well conserved, at least up to 48 hours.

1. INTRODUCTION

In the barotropic forecasts made by the Joint Numerical Weather Prediction Unit (JNWP) the absolute vorticity is assumed to be conserved, or:

$$\frac{d\eta}{dt}=\frac{d}{dt}(\nabla^2\psi+f)=0 \tag{1}$$

where η is the absolute vorticity, ψ is the 500-mb. stream function, and f is the Coriolis parameter. At JNWP the initial stream-function values used in the barotropic forecast are obtained using a method developed by Shuman [1] for solving the "balance equation,"

$$\nabla^2\phi=f\nabla^2\psi+2\left[\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{dy^2}-\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2\right]+\frac{\partial\psi}{\partial y}\frac{\partial f}{\partial y}+\frac{\partial\psi}{\partial x}\frac{\partial f}{\partial x} \tag{2}$$

where ϕ is the geopotential. In actual practice, the relative vorticity is approximated by the finite-difference vorticity or

$$\nabla^2\psi\approx\mathbb{V}^2\psi=\frac{m^2}{d^2}(\psi_1+\psi_2+\psi_3+\psi_4-4\psi.) \tag{3}$$

where $\mathbb{V}^2$ is the finite-difference Laplacian, m is the map factor, and d is the mesh length.

Since a method for obtaining exact solutions for most of the differential equations used in describing atmospheric processes is not known, one customarily resorts to the use of finite-difference techniques. The resultant approximations introduce errors called "truncation errors." Examination of cases wherein serious errors resulted in the barotropic forecasts made during the winter of 1956–57 led one to wonder if the absolute vorticity really was conserved. Perhaps truncation errors grew sufficiently to obscure the real physical developments. Even though the differential form of the vorticity equation expresses conservation, the finite-difference form may not insure it.

Substitution of the approximate equation (3) in (1) gives

$$\frac{d}{dt}\left[\frac{m^2}{d^2}(\psi_1+\psi_2+\psi_3+\psi_4-4\psi)+f+R\right]=0 \tag{4}$$

where R is the quantity which must be added to the right

[1] On temporary assignment to JNWP.
[2] Any opinions expressed by Lcdr. Hubert are his own and do not necessarily reflect the views of the Navy Department at large.

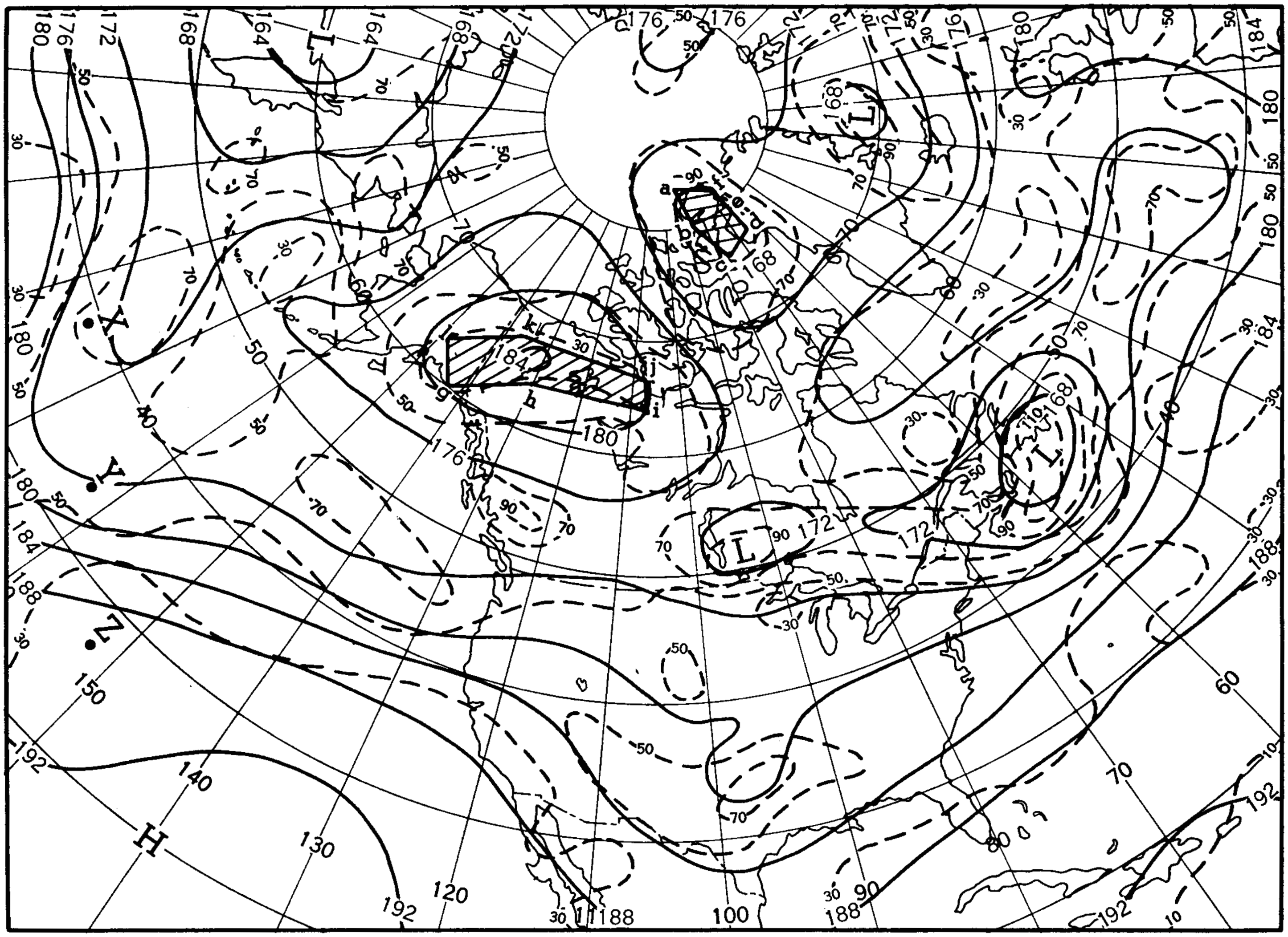

FIGURE 1.—The initial 500-mb. chart, 0300 GMT March 5, 1957. Solid lines are contours in hundreds of feet; dashed lines are absolute vorticity in units of 8×10^{-3} hr^{-1}. Trajectories were computed for points X, Y, and Z.

side of equation (3) to make it exact. Thus the finite-difference form of the vorticity equation becomes

$$\frac{d}{dt}(\nabla^2\psi+f)+\frac{dR}{dt}=0 \tag{5}$$

That is, in the finite-difference form the absolute vorticity is conserved only if dR/dt is equal to zero. The objective of this study was, therefore, to determine if the truncation errors are sufficiently large to make the absolute vorticity in the barotropic model nonconservative for forecasting purposes.

2. TRAJECTORIES OF PARCELS

During the progress of the barotropic forecast, fields of hourly 500-mb. stream-function values are stored on a "history tape." From these stored fields 72-hour trajectories of individual air parcels may be computed using a method due to Hubert [2]. The winds which are used in the trajectory computation are obtained by differentiating (at the location of the particle) a least squares cubic surface fit to 16 stream-function values surrounding the parcel being tracked. As a test of the accuracy of the cubic method, Hubert advected several parcels in a field held constant with time and found that the stream-function values along the trajectory departed from their initial values by only slight amounts over the course of several days.

Even though the trajectories are based upon predicted stream-function values which have been obtained from a barotropic forecast, they can be used to determine whether or not the finite-difference wind advects absolute vorticity at or near the same speed as the more accurate wind. For example, if it should be found that the absolute vorticity changes appreciably along the trajectory, one could argue that truncation errors really are serious.

Figure 1 shows the initial 500-mb. analysis for 0300 GMT March 5, 1957, and figures 2–4 the 24-, 48-, and 72-hour barotropic forecasts made from the initial chart. Points X, Y, and Z of figure 1 are three arbitrarily selected points for which trajectories were computed. The forecast locations of these points after 24, 48, and 72 hours,

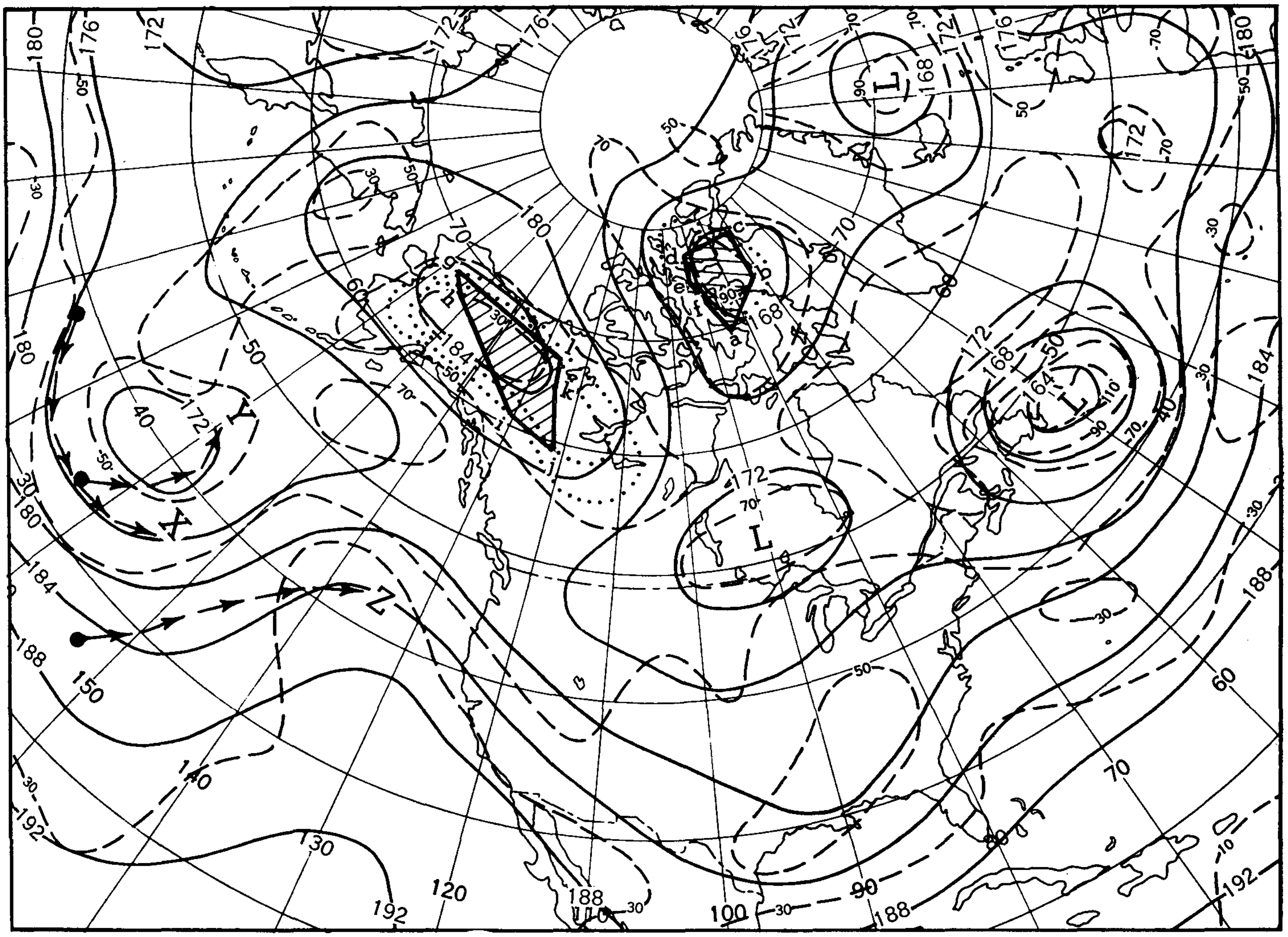

FIGURE 2.—Barotropic 500-mb. forecast for 24 hours from 0300 GMT March 5, 1957. Arrows show 24-hour trajectories of X, Y, and Z.

as well as the paths they followed, are shown in figures 2, 3, and 4. The absolute vorticity values for each field are indicated by dashed lines.

A summary of the initial and forecast absolute vorticities at 24-hour intervals along each trajectory is presented in table 1. Examination of the results reveals that the absolute vorticities of the points were fairly well conserved, but not perfectly. Another way to look at the error is to find how far the end points of the trajectories come from points in the grid where the absolute vorticities were within, say, 10 percent of the initial values. The "error" columns of table 1 show that even at 72 hours no point is off by as much as one mesh length (3° of latitude at 45° N.).

TABLE 1.—*Observed and forecast absolute vorticities in units of 8×10^{-3} hr^{-1} for parcels X, Y, and Z. Percentage change from initial absolute vorticities is also given. The number in the "error" column shows the distance, in mesh length, the end point of the trajectory would have to be displaced to make the absolute vorticity error 10 percent or less*

Point	March 5	March 6			March 7			March 8		
	η	η	Percent change	Error	η	Percent change	Error		Percent change	Error
X	76	70	8	0.0	57	25	0.7	65	14	0.9
Y	70	56	20	.4	60	14	.5	60	14	.9
Z	21	21	0	.0	30	43	.8	25	19	.4

3. CONSERVATION OF AREAS OF ABSOLUTE VORTICITY

The second experiment carried out in this study was somewhat different in that it dealt with the conservation of areas. In the JNWP barotropic atmosphere the divergence is zero so that the Area A of a closed curve on the earth's surface is conserved or

$$\frac{1}{A}\frac{dA}{dt}=\nabla\cdot\mathbf{V}=0 \tag{6}$$

If one selects and follows a closed curve along which the absolute vorticity is constant, then in a barotropic atmosphere the area of the curve should remain unchanged from day to day and it should lie along the same absolute vorticity isoline.

Consider the polygon abcdef (fig. 1) which is located

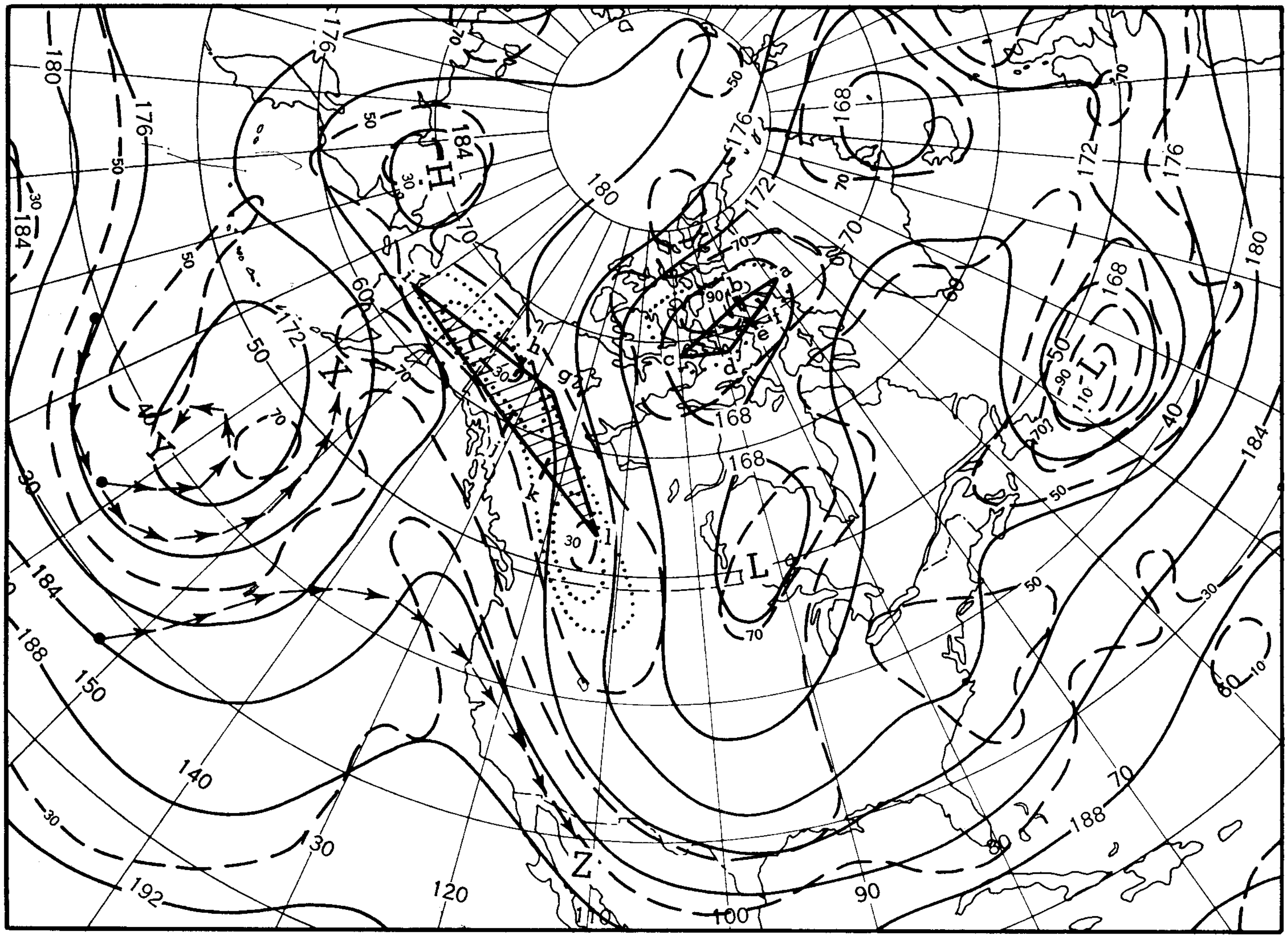

FIGURE 3.—Barotropic 500-mb. forecast for 48 hours from 0300 GMT March 5, 1957. Arrows show 48-hour trajectories of X, Y, and Z.

near latitude 75° N., longitude 75° W. at 0300 GMT March 5, 1957. The vertices of the polygon have been selected on the absolute vorticity isoline of 90, which in this case encircles a cyclonic maximum. (In order to simplify machine programing and computation, the absolute vorticity is expressed in units of 8×10^{-3} hr^{-1}.) Hourly positions of each of the points defining the polygon were computed with the trajectory code. By connecting the end points of the 24-hour trajectories for the six points, one obtains the location and shape of the polygon at the end of this period (see fig. 2). Similarly, the locations and shapes after 48 and 72 hours are given in figures 3 and 4. The areas of the polygons were measured, corrections were made for map scale, and the percentage change in areas computed. The results are shown in table 2.

TABLE 2.—*Area (in square inches) of polygon abcdef, and the percentage change from the initial area*

	Mar. 5	Mar. 6	Mar. 7	Mar. 8
Area	1.12	1.22	0.89	1.07
Percent change		8	12	4

The area abcdef at the end of 72 hours differed from its original size by only 4 percent. This difference is within the limits of measurement, and the table shows that the computational errors were not sufficient to change the area of polygon abcdef by very much. It is also possible to get some idea of the changes in absolute vorticity from figures 2, 3, and 4.

Figure 2 shows that all the points a, b, c, d, e, and f fell between the absolute vorticity isolines of 85 and 90. (The original value was 90.) In figure 3 all the points fell between the 80 and 90 isolines, and in figure 4 all the points fell within the 80 line. However, in the latter figure the 90 isoline has completely disappeared so that the area corresponding to this particular initial value has not been strictly conserved. On the other hand, at two grid points contained within the 72-hour polygon the absolute vorticity values were 85 and 87.

Let us now consider the polygon ghijkl, located near 65° N., 130° W. in figure 1. These vertices lie on the absolute vorticity isoline of 30 which encircles a minimum of absolute vorticity. Even though the polygon underwent

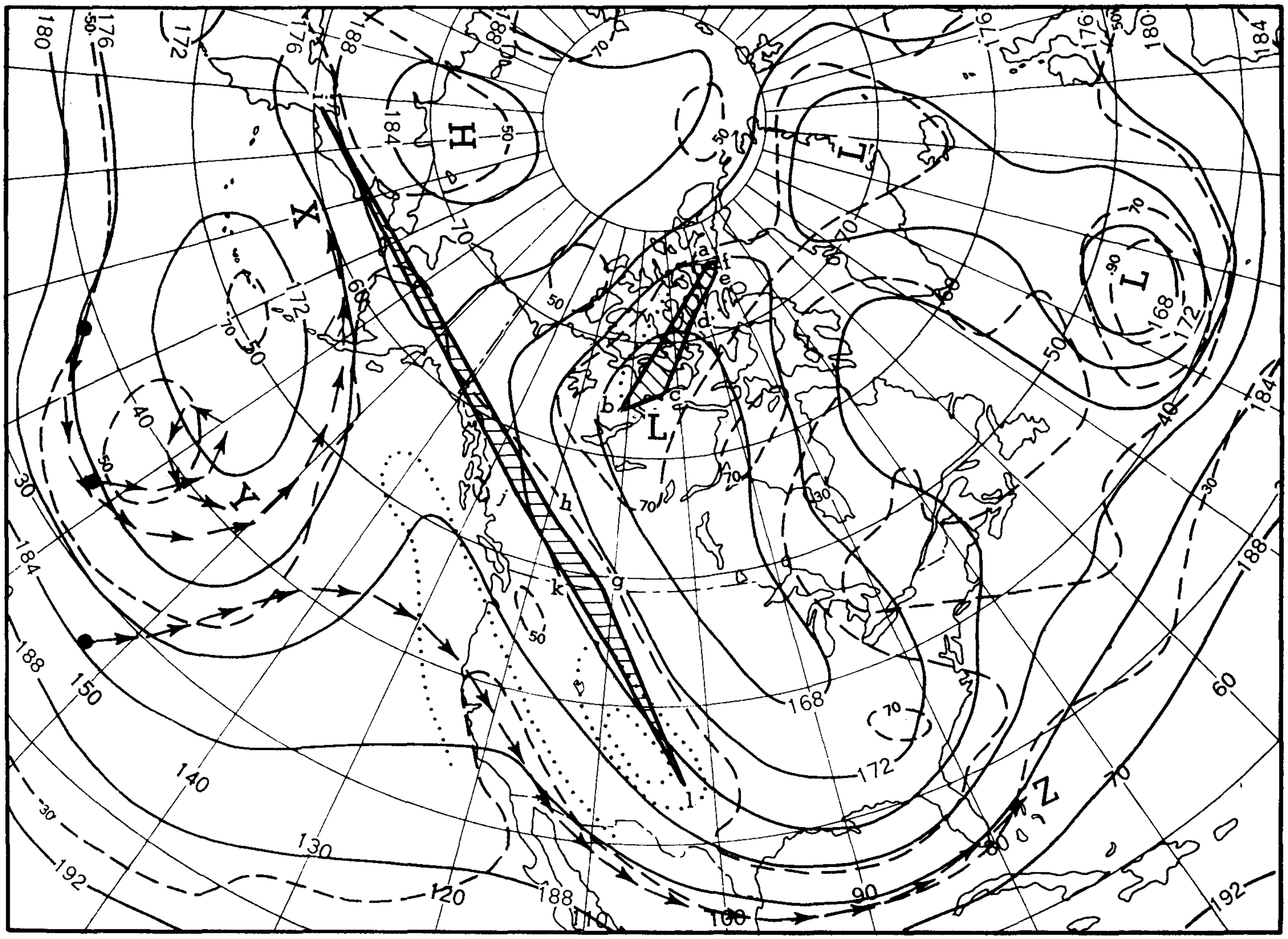

FIGURE 4.—Barotropic 500-mb. forecast for 72 hours from 0300 GMT March 5, 1957. Arrows show 72-hour trajectories of X, Y, and Z.

considerable displacement and deformation in the next 72 hours (figs. 2, 3, 4) the area was conserved to within 9 percent of its initial value. The results in this case are shown in table 3.

Referring to figure 2, one sees that except for point g the entire polygon fell inside the 40 absolute vorticity isoline. At 48 hours (fig. 3) the polygon was rather distorted, and at the same time the absolute vorticity pattern had two separate centers with minimum values near 30. By 72 hours (fig. 4) the polygon had been stretched almost beyond recognition; yet, the enclosed area differed from the original by only 7 percent. The 30 isoline had vanished from the neighborhood of the polygon, however, and in this case the absolute vorticity was not conserved for 72 hours. The mean value of the absolute vorticity over the polygon had increased to about 48.

TABLE 3.—*Area (in square inches) of polygon ghijkl, and percentage change from initial area*

	Mar. 5	Mar. 6	Mar. 7	Mar. 8
Area	3.39	3.11	3.08	3.64
Percent change		8	9	7

One gets the impression that the initial absolute vorticity of the second polygon was stretched out into such a long, narrow band that it was lost between grid points. In the course of the barotropic forecast, the field of $\nabla^2\psi$ is smoothed every 12 hours to eliminate the shorter wavelengths. This has the effect of wiping out narrow filaments of vorticity. Synoptic experience suggests that nature acts in much the same way; thin elongated ridges at the 500-mb. level have short life periods. Welander [3] has described the development of a deformation pattern quite similar to that obtained in the case of polygon ghijkl.

4. ANTICYCLOGENESIS OF MARCH 1, 1957

One of the characteristics of the 500-mb. barotropic forecasts during the winter of 1956–57 was excessive anticyclogenesis in mid-Pacific. The 48-hour forecast made from initial data for 0300 GMT March 1, 1957, gave a High with a central value of 19,600 ft. near 42° N., 153° W.; by 72 hours the central value had risen to 19,900 ft. and retrogression toward the west continued. These values turned out to be about 2000 ft. higher than

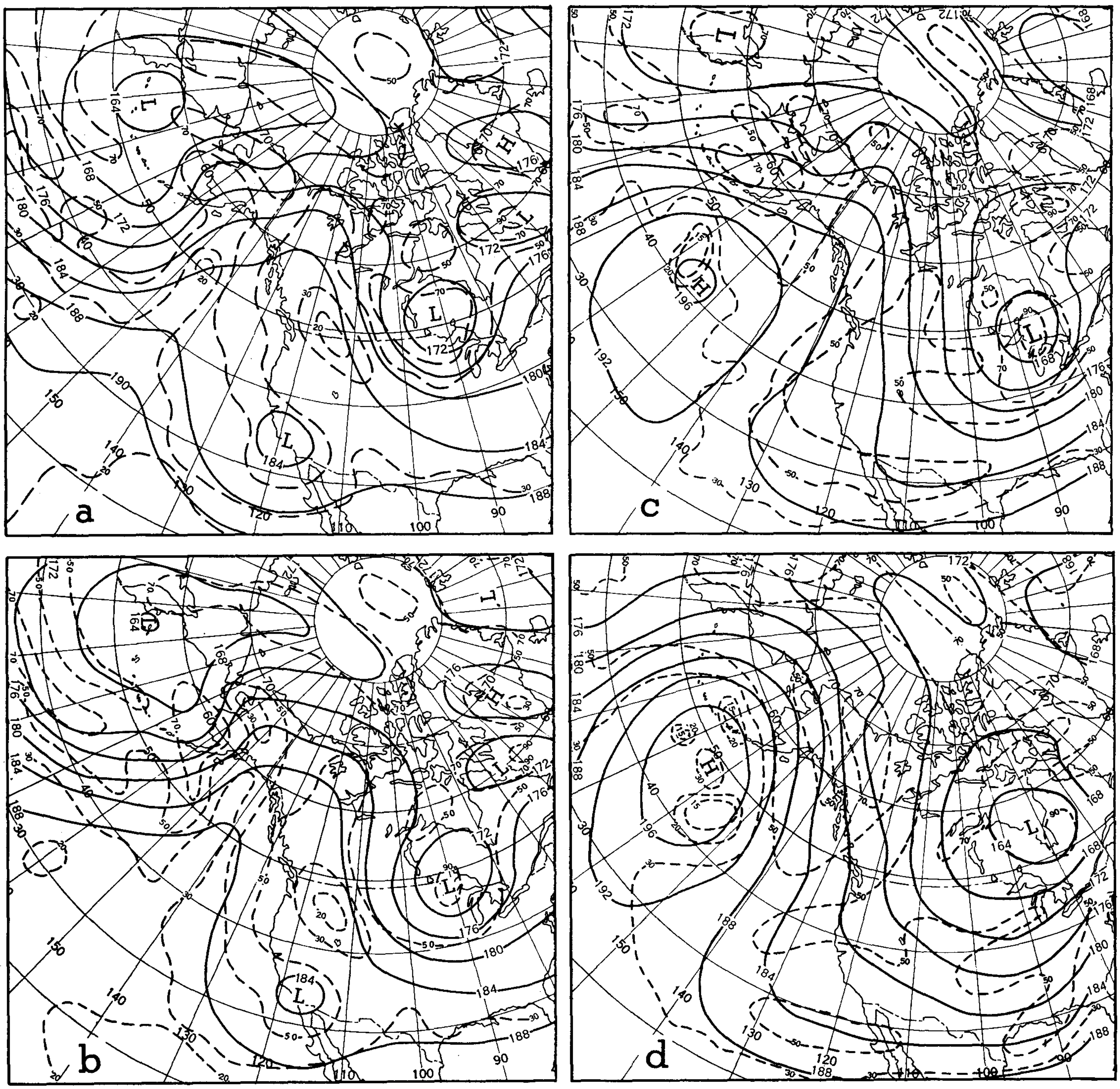

Figure 5.—500-mb. barotropic forecasts from 0300 GMT March 1, 1957: (a) for 12 hours, (b) for 24 hours, (c) for 48 hours, and (d) for 72 hours. Solid lines are contours in hundreds of feet. Dashed lines are absolute vorticity in units of 8×10^{-3} hr^{-1}.

observed. Did truncation errors play an important role in the excessive anticyclogenesis?

Figure 5 shows the sequence of forecast maps for this case while figure 6 shows the time variation of the lowest value of absolute vorticity that was moving into the neighborhood of the anticyclone's center. The changes in the minimum absolute vorticity indicate that strict conservation did not occur here. The 12-hour forecast chart had a minimum of 18, but values as low as 11 could be found in the 48-hour forecast and 14 at 72 hours. Truncation errors are offered as a partial explanation for this observed decrease in absolute vorticity. However, it does not appear that these errors were sufficiently large or spread over a great enough area to account for much of the anticyclogenesis. Rather, it seems that errors due to unrealistic boundary assumptions may have been of prime importance because there was strong inflow across the southern boundary throughout the forecast period.

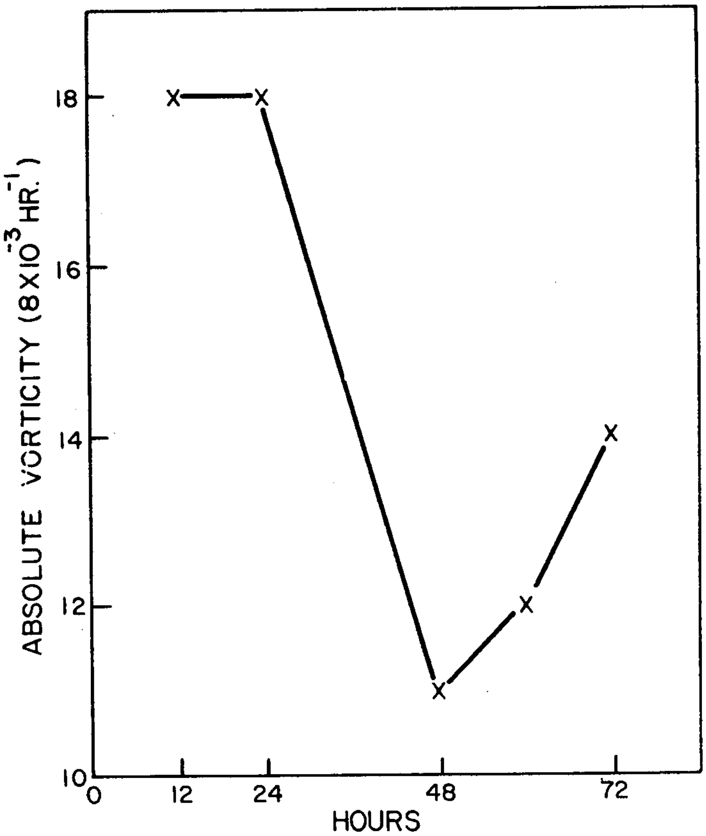

FIGURE 6.—Change in minimum absolute vorticity entering the Pacific anticyclone. Based on barotropic forecasts make from 500-mb. data for 0300 GMT March 1, 1957.

Experiments at JNWP have shown that forecast anticyclogenesis can be reduced in this area by shifting the grid so that flow across the boundaries is minimized (Cressman and Hubert [4]).

5. CONCLUSIONS

The tests conducted in the course of this study showed conservation of area together with small changes in absolute vorticity along the cubic trajectories. This limited evidence seems to indicate that truncation errors due to the use of finite-difference methods are not too serious in the barotropic forecasts, especially out to 48 hours.

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